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On the Demonstration of Formulæ connected with Interest and Annuities. *By* PROFESSOR DE MORGAN.

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IN most branches of mathematics, the actual use of fundamental processes in the form which the first definitions suggest, is often supplanted, either by processes of greater skill, or by the use of pure reasoning. In the subject of which this paper treats, there has not been much attempt to connect formulæ by reasoning. The actual exhibition of successive annual results has been the only method extensively employed; and it throws the required total result into a series of terms: this series is either algebraically summed, or calculated term by term for insertion in a table.

The present paper is intended to show that this summation of algebraical series may be dispensed with, at least in questions of annuities certain: and also that common points of principle, which the ordinary methods leave altogether out of sight, will reduce many questions of life annuities to an absolute coincidence of form with the corresponding questions of ordinary annuities.

An annuity, by definition, is a succession of payments to be made at the end of periods determined by the problem. In the ordinary question, the periods of payment are those at which money returns its interest. An annuity of £1 for *n payments* means that £1 is to be paid by the grantor at the end of each of the first *n* periods at which money now put out receives interest. The periods need not be years; the first may be five minutes, the second a

revolution of Neptune, the third the duration of a life to be put in at the end of the second, and so on. It is enough that the grantee has no right to call upon the grantor until the latter has just received a term's interest upon his outstanding capital.

Using the small letters r and v when the terms are years, let R and V refer to the cases in which the term is any status whatsoever, certain or contingent; the rule being that the second status begins at the moment the first ends. Then $1+R$ is what 1 becomes at the end of the status, and V the present value of 1 to be received when the status ends; hence $V=(1+R)^{-1}$ and V^n is the present value of 1 to be received at the end of the nth status. A person who now gives 1 gives the equivalent of R, R, R, &c. for ever, payable at the end of the first, second, &c. status; or what, in a wide sense, we must call a perpetual *annuity*, R, R, R, &c. A person who gives 1 now, and revokes his gift at the end of the nth status, gives n terms of R, R, R, &c. Let him discount his revocation—that is, let him give only $1-V^n$, and he gives n terms of R, R, R, &c. That is, $(1-V^n)R^{-1}$ is the present value of n terms of 1, 1, 1, &c.

Let the status be a life: it may be strictly a life, or a life with a given term added, or an indefinite term, as to the end of the year of death; or any other, according as V is taken. Then V is the present value of a reversion of 1, and $(1-V^n)R^{-1}$, or $(V-V^{n+1})(1-V)^{-1}$, is the value of all the fines, each fine being 1, upon n successive lives of equal value, the first of which is just put in; and the value of the fines for ever is R^{-1} or $V(1-V)^{-1}$. This formula was given by Mr. Peter Gray, in an excellent paper on successions in a former Number of this *Journal;* Mr. Milne's formula is $V(1-v^{t+1})^{-1}$, where t is the number of years certain which the annuity on the life is worth. The connection involves the following theorem:—If the life be worth t years certain, the reversion of £1 is worth £1 due at the end of $t+1$ years; for if $A=(1-v^t)r^{-1}$, it is plain that $(1-rA):(1+r)$, or $(1-rA)v$, is $v^t\times v$, or v^{t+1}. That this should have escaped Milne, is to be attributed to the total want of reasoning on the connection of formulæ which had existed up to his time. His method is an independent one, produced by the error which had prevailed on the subject of successive lives. Mr. Ryley mentioned to me the formula $V(1-V)^{-1}$, produced by him from the series $V+V^2+$, &c., and obtained by seeing that V, V^2, &c. are the values of the successive fines, just at the time when I had put by for consideration the remark that every V must have its R, even though V stand for the value of a contingent

reversion, and that every formula of annuities certain must mean *something* in life contingencies.

If the rates of improvement vary from term to term—that is, if during the nth status 1 become $1+R_n$, we may allow $V^{(n)}$ to represent $V_1 . V_2 . V_3 \ldots\ldots V_n$. All formulæ may be written as now, on the understanding that *powers* of V become successive products. Thus a perpetual annuity of £1 is $V^{(1)} : (1-V^{(1)})$, considered as representing $V^{(1)}+V^{(2)}+V^{(3)}+$ &c., or $V_1+V_1V_2+V_1V_2V_3+\ldots$ After this remark, there is no further occasion to consider unequal rates of improvement.

The present value of an annuity of £1 *per annum*, continued until the extinction of the nth life, is $r^{-1}-(1+r^{-1})V^n$, or $(1+r^{-1})(1-V^n)-1$; since V^n is the present value of £1 deferred till the end of the year of such extinction. This may easily be verified by showing that the value of a fine of $(1+r)(1+A) : (1-rA)$ at each death is $(1+r^{-1})(1-V^n)$. The deduction is the present pound which enters the premium, but not the annuity.

When V is the value of a reversion of £1, or $1+R=(1+r) : (1-rA)$, we find, by making $A=(1-v^t) : r$, that $1+R=(1+r)^{t+1}$, as we ought to expect. The substitution of an annuity certain for a life annuity affords a means of verifying formulæ, and may lead to the habit of observing analogies.

Any number of years of the perpetual annuity A, B, C, D, &c. is easily converted into another perpetual annuity. Thus the five years A, B, C, D, E, are equivalent to the perpetual annuity $A-Fv^5$, $B-Gv^5$, $C-Hv^5$, $D-Kv^5$, $E-Lv^5$, $F-Mv^5$, &c.; for the withdrawals of Fv^5, Gv^5, &c., are compensated by the subsequent payments of F, G, &c., leaving nothing uncompensated except A, B, C, D, E, at the ends of the first five years. Hence, also, n years of 1, 1, 1, &c. is a perpetual annuity of $1-v^n$, or $(1-v^n)r^{-1}$.

The accumulations of an annuity can also be converted into a perpetual annuity. Thus the perpetual annuity $Av^{-3}-D$, $Bv^{-3}-E$, $Cv^{-3}-F$, $Dv^{-3}-G$, $Ev^{-3}-H$, &c., is obviously only the annuity Av^{-3}, Bv^{-3}, Cv^{-3}, which is but the *terminal value* of A, B, C, or what arises from its accumulation. Thus the terminal value of n years of 1, 1, 1, &c. is a perpetual annuity of $v^{-n}-1$ or $(1+r)^n-1$, as commonly given. Milne has well obtained this result by observing that the compound interest of £1 in n years can only be the accumulations of n years of r, r, r, &c.

Again, $\{(1+R)^n-1\}R^{-1}$, if $(1+R)^{-1}$ or V be the value of an assurance of £1 on a certain life, represents the last amount assured,

as follows :—There is a fine of £1 at each of n successive deaths, and the receiver invests the first fine in an assurance on the second life, which assurance, with the second fine, he invests in an assurance on the third life, and so on.

The powers of r^{-1} represent perpetual annuities, as follows :— In r^{-2} we see the perpetual annuity r^{-1}, r^{-1}, r^{-1}, &c., or the annuities 0, 1, 1, 1, &c., 0, 0, 1, 1, &c., 0, 0, 0, 1, &c., or the single annuity 0, 1, 2, 3, &c.; or 1, 2, 3, &c., deferred one year. Similarly, r^{-3} is the perpetual annuity r^{-2}, r^{-2}, r^{-2}, &c., or the sum of 0, 0, 1, 2, 3, &c., 0, 0, 0, 1, 2, &c., 0, 0, 0, 0, 1, &c., or the annuity 0, 0, 1, 3, 6, &c.; or 1, 3, 6, &c., deferred two years. Generally, p_q standing for the number of combinations of p out of q, r^{-n} is the value of the annuity 1, 1_n, 2_{n+1}, 3_{n+2}, &c., deferred $n-1$ years. Hence $\dfrac{(1+r)^{n-1}}{r^n}$ = present value of 1, 1_n, 2_{n+1}, 3_{n+2}, &c.

Any formula which has a power of r in the denominator may be reduced by interpretation to its perpetual annuity. For example, $r^{-3}-(v+v^2)r^{-2}+r^{-1}$. Here we have

r^{-3} =	0,	0,	1,	3,	6,	10,	15,	21,	&c.
r^{-1} =	1,	1,	1,	1,	1,	1,	1,	1,	&c.
$-vr^{-2}$ =	0,	0,	-1,	-2,	-3,	-4,	-5,	-6,	&c.
$-v^2r^{-2}$ =	0,	0,	0,	-1,	-2,	-3,	-4,	-5,	&c.
Annuity required	1,	1,	1,	1,	2,	4,	7,	11,	&c.

To find the value of n years of 1, 2, 3, &c.—Make a grant of £1 now, and of an annuity 1, 1, 1, &c., of $n-1$ years; which we may represent by 1 | 1, 1, 1, &c. Reclaim the n pounds at the end of n years. This amounts to granting an annuity of r, $2r$, $3r$, ... nr. The grants are worth $1+(1-v^{n-1})r^{-1}$, and the revocation worth nv^n; whence $1+(1-v^{n-1})r^{-1}-nv^n$ is the value of r, $2r$, $3r$, ... nr, or $\dfrac{1+r}{r^2}-\dfrac{rv^{n-1}+nv^n}{r^2}$ = present value of 1, 2, 3, ... n.

This method might be carried further, as by making the grant 1 | 2, 3, ... n; and so on.

Let P_{m-1} signify the sum of m terms of the development of $(1+r)^n$, or $1+1_nr+\ldots.+(m-1)_nr^{m-1}$. Consider the fractions

$$\frac{1-P_0v^n}{r},\quad \frac{1-P_1v^n}{r^2},\quad \frac{1-P_2v^n}{r^3},\quad \frac{1-P_3v^n}{r^4},\ \&c.,$$

which call Q_0, Q_1, Q_2, Q_3, &c. We have then

$$Q_1=\frac{Q_0-1_nv^n}{r},\quad Q_2=\frac{Q_1-2_nv^n}{r},\quad Q_3=\frac{Q_2-3_nv^n}{r},\ \&c.$$

Here Q_0 is n years of 1, 1, 1, &c., and Q_1 is a perpetual annuity of $Q_0 - 1_n v^n$. For simplicity, take $n=6$. First, Q_0, Q_0, Q_0, &c. is 0, 1, 2, 3, 4, 5, 6, 6, 6, &c., and $1_6 v^6$, $1_6 v^6$, &c. is 0, 0, 0, 0, 0, 0, 6, 6, 6, &c.; whence Q_1 is 0, 1, 2, 3, 4, 5, or 1, 2, 3, 4, 5, deferred one year. Generally, Q_1 is always $n-1$ years of 1, 1_2, 2_3, 3_4, &c., deferred one year. Again, $2_6=15$, and Q_2 is a perpetual annuity of $Q_1 - 15v^6$. But Q_1, Q_1, Q_1, &c. is the sum of the annuities 0, 0, 1, 2, 3, 4, 5 and 0, 0, 0, 1, 2, 3, 4, 5, &c., or 0, 0, 1, 3, 6, 10, 15, 15, 15, &c.; and $15v^6$, $15v^6$, &c. is 0, 0, 0, 0, 0, 0, 15, 15, &c.: whence Q_2 is 0, 0, 1, 3, 6, 10, or four years of 1, 3, 6, 10, deferred two years. Generally, P_2 is always $n-2$ years of 1, 1_3, 2_4, 3_5, &c., deferred two years; and thus it may be shown that Q_m is the value of $n-m$ years of 1, 1_{m+1}, 2_{m+2}, 3_{m+3}, &c., deferred m years. Multiplying by $(1+r)^m$, we find that $(1+r)^m r^{-(m+1)}$ $(1-P_m v^n)$ is the value of $n-m$ years of 1, 1_{m+1}, 2_{m+2}, &c.; or, the first $n-m$ years of the perpetual annuity 1, 1_{m+1}, 2_{m+2}, &c., are worth the fraction $1-P_m v^n$ of the whole.

A corresponding theorem on *terminal values*, or *accumulations*, may be independently deduced as follows:—Let

$$Q_0 = \frac{(1+r)^n - P_0}{r}, \quad Q_1 = \frac{(1+r)^n - P_1}{r^2}, \quad Q_2 = \frac{(1+r)^n - P_2}{r^3}, \text{ \&c.}$$

Let such a symbol as A, B, C | D, E, F, &c. denote the value of the past and future in the annuity on which C has just been paid. In Q_0 we see the terminal value of n years of 1, 1, 1, &c., and

$$Q_1 = \frac{Q_0 - 1_n}{r}, \quad Q_2 = \frac{Q_1 - 2_n}{r}, \quad Q_3 = \frac{Q_2 - 3_n}{r}, \text{ \&c.}$$

Let $n=6$. Q_1 is a perpetual annuity of $Q_0 - 6$, of which Q_0, Q_0, &c. may be resolved into 1, 1, 1, 1, 1 | 1, 1, 1, 1, 1 | 1, 1, &c., with a total of 1, 2, 3, 4, 5 | 6, 6, 6; and the withdrawal of 6, 6, 6, &c. leaves 1, 2, 3, 4, 5 |, the terminal value of five years of 1, 1_2, 2_3, 3_4, &c. Again: Q_2 is a perpetual annuity of $Q_1 - 2_6$, or $Q_1 - 15$, of which Q_1, Q_1, &c. is 1, 2, 3, 4 | 5 and 1, 2, 3 | 4, 5, &c., with a total of 1, 3, 6, 10 | 15, 15, &c. This the removal of 15, 15, &c. reduces to 1, 3, 6, 10 |, the terminal value of four years of 1, 1_3, 2_4, 3_5, &c. Proceeding in this way, we show that Q_m or $\{(1+r)^m - P_m\} r^{-(m+1)}$ is the terminal value of $n-m$ years of 1, 1_{m+1}, 2_{m+2}, &c.

I need not now say, that the values of successions of variable fines may be found by the same formulæ, and on the same mode of reasoning. The fact is, that a *contingency* neither does nor can

enter into any formula whatever. That which a formula calculates is a certainty; and with the act of the mind by which the certainty is accepted in lieu of a contingency, and the nature of things by which events come to justify such acceptance, algebra and its processes have nothing to do. Because the probability of an event is *three tenths*, we hold *three tenths* of the benefit which such event will bring with it if it should happen, to be the equivalent certainty. We may be right or wrong, and we know that we are right; but, right or wrong, the assignment of $av+bv^2+cv^3+$ &c. as the value of a life annuity is the assignment of the annuity certain a, b, c, &c. as a composition for the contingency.

We are in the habit of saying that the value of an event multiplied by the chance of its happening is the *value of the chance*: it would be better, in some respects, to call this product the *equivalent certainty*. Such a phrase, had it been adopted from the beginning, would perhaps have prevented that utter separation which has taken place between the cases in which $av+bv^2+$ &c. represents a life annuity, and that in which it represents an annuity certain.

Among the preceding solutions is that of a problem proposed in the last Number (p. 243). One solution was forwarded to the Editor; but it did not fulfil the required conditions. For instance, the summation of the following "series involving powers," $1+r+\ldots+r^{n-1}$, was effected by multiplying it by $1-r$, and thus reducing it to $1-r^n$. Between this and the theorem $(1-r^n):(1-r)=1+r+\ldots+r^{n-1}$ there is but difference of form.

On a simple plan of Classifying the Policies of a Life Assurance Company, so as to possess, at any time, the means of forming a Table of the Mortality experienced in the Office. By SAMUEL BROWN, F.S.S., *One of the Honorary Secretaries of the Institute of Actuaries, and Actuary of The Mutual Life Assurance Society.*

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AFTER the valuable essays with which the members of this Institute have been favoured by our learned Vice-Presidents and other gentlemen on the subjects of our professional studies, I feel that an apology is due to you for introducing a topic of apparently so humble a character; but I have been gratified to perceive, that many of the papers read before you, and especially those above referred to, have been of such a practical character as to prove